EBERHARD KARLS UNIVERSITÄT TÜBINGEN

METHODS CENTER

**Wirtschafts- und Sozialwissenschaftlichen Fakultät**
Methodenzentrum

# Toward a unified perspective on assessment models

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# Outline

# Introduction

- For details see, Noventa, S., Heller, J., Kelava, A. (2024). Toward a unified perspective on assessment models, part I: foundation of a framework. *Journal of Mathematical Psychology*, 122, 102872.
- Part II (dichotomous variables) and Part III (polytomous and continuous variables) are still under review
- Theories of assessment have different traditions in Psychometrics and Mathematical Psychology with related yet different scopes and aims.

1. Item Response Theory (IRT)
   a) parametric and non-parametric (NIRT)
   b) uni- and multi-dimensional (MIRT)
2. Cognitive Diagnostic Models (CDMs) or Assessment (CDA)
   a) ‘early’ CDA approaches, single models
   b) ‘modern’ CDA approaches, general models and frameworks
3. Knowledge Space/Structure Theory (KST)
   a) knowledge/performance-based approach
   b) competence-based approach (CbKST)

- These traditions have generated a plethora of models.

# Introduction

| Acronym | Definition | Acronym | Definition |
|---|---|---|---|
| **KST** | Knowledge Space/structure Theory | **IRT** | (parametric) Item Response Theory |
| BLIM | Basic Local Independence Model | *n*PL | *n*-Parameters Logistic |
| CBLIM | Competence based BLIM | *n*PNO | *n*-Parameters Normal Ogive |
| LKS | Logistic Knowledge Structure | 1PL-AG | 1PL- Ability-based Guessing |
| SLM | Simple Learning Model | 2PLE | 2-Parameters Logistic Extension |
| **CDA** | Cognitive Diagnostic Assessment | ARRG | Ability Removing Random Guessing |
| DINO | Deterministic Input Noisy OR-gate | LLTM | Linear Logistic Test Model |
| RDINO | Reparametrized DINO | GRM | Graded Response Model |
| DINA | Deterministic DINA | BNM | Bock's Nominal Model |
| RDINA | Reparametrized DINA | RSM | Rating Scale Model |
| HO-DINA | Higher-Order DINA | PCM | Partial Credit Model |
| MS-DINA | Multiple Strategy-DINA | GPCM | Generalized PCM |
| GDINA | Generalized DINA | CRMs | Continuation Ratio Models |
| MCLCM-C | Multiple Classification Latent | **MM-IRT** | Mixture-Measurement IRT |
| | Class Model - Conjunctive | MRM | Mixed Rasch Model |
| MCLCM-D | Multiple Classification Latent | **NIRT** | Non-parametric IRT |
| | Class Model - Disjunctive | MHM | Monotonic Homogeneity Model |
| NIDO | Noisy Input Deterministic-Or-gate | DMM | Double Monotonicity Model |
| NIDA | Noisy Input Deterministic AND-gate | **MIRT** | Multidimensional IRT |
| RUM | Re-parametrized Unified Model | M*n*PL | Multidimensional *n*PL |
| NC-RUM | Non-Compensatory RUM | MGRM | Multidimensional GRM |
| C-RUM | Compensatory RUM | MPCM | Multidimensional PCM |
| R-RUM | Reduced RUM | MGPCM | Multidimensional GPCM |
| LLM | Linear Logistic Model | M2PL-AIG | Multidimensional 2PL |
| ACDM | Additive-Cognitive Diagnostic Model | | Ability-based Item Guessing |
| GDM | General Diagnostic Model | GPMNM | General-Purpose Multidimensional NM |
| MGDM | Mixture distribution GDM | MLTM | Multicomponent Latent Trait Model |
| ECDM | Explanatory Cognitive Diagnostic Model | MLTM-D | MLTM for Diagnostic |
| LCDM | Log-linear Cognitive Diagnostic Model | GLTM | General component Latent Trait Model |
| PINC | Probabilistic Input, Noisy Conjunctive | MNO | Multidimensional Normal-Ogive |
| HO-PINC | Higher-Order PINC | M3NO | Multidimensional 3-parameters NO |

# Introduction

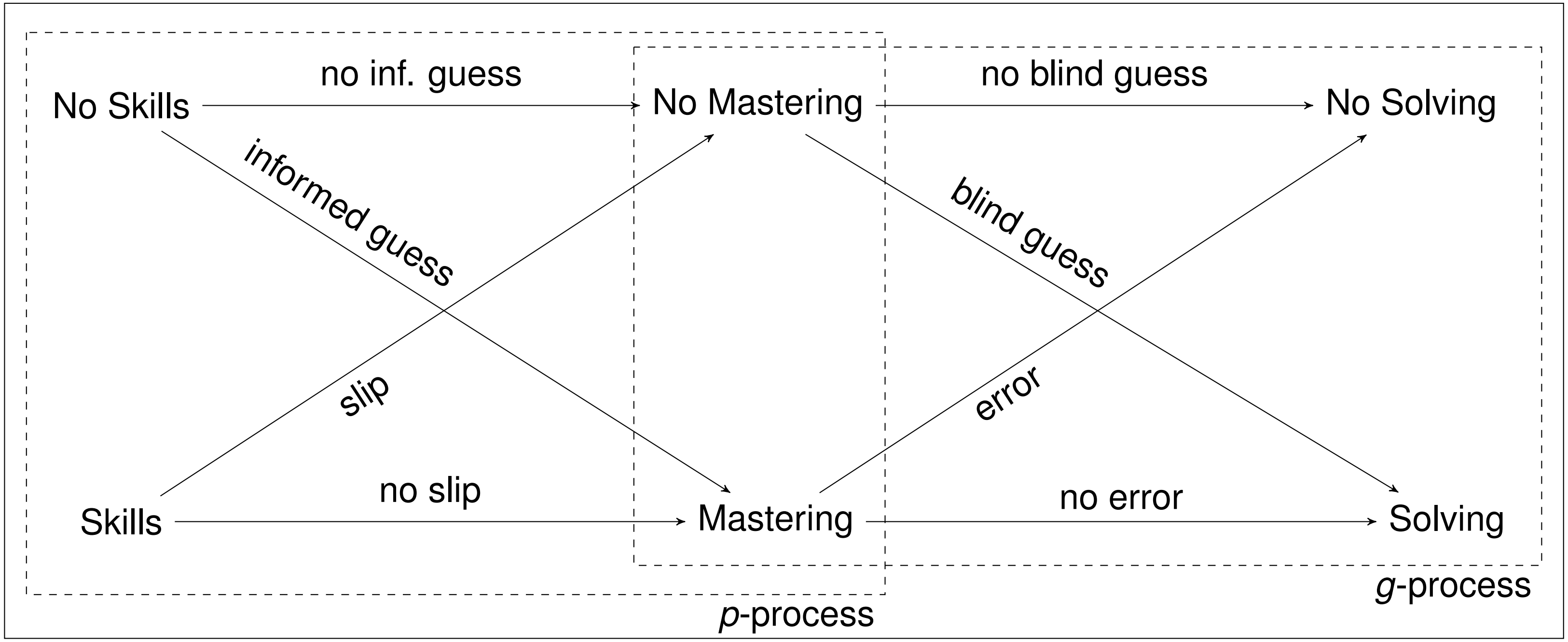
Similarities and connections have been highlighted, especially with Latent Class Analysis (LCA). Some examples

a) betw. IRT and LCA (Formann, 1982, 1985)
b) betw. CDA and LCA (Maris, 1999)
c) betw. CDA and IRT (Junker & Sijtsma, 2001)
d) betw. KST and LCA (Schrepp, 2005; Ünlü, 2006, 2011)
e) betw. KST and Mokken's NIRT model (Ünlü, 2007)
f) betw. KST and IRT (Stefanutti, 2006; Noventa et al., 2019)
g) betw. KST and Bayesian networks (Burigana, 2024)
h) betw. CbKST and CDA (Heller et al., 2015)

**Can we identify a set of common primitives and operations that allows to systematize and derive these models?**

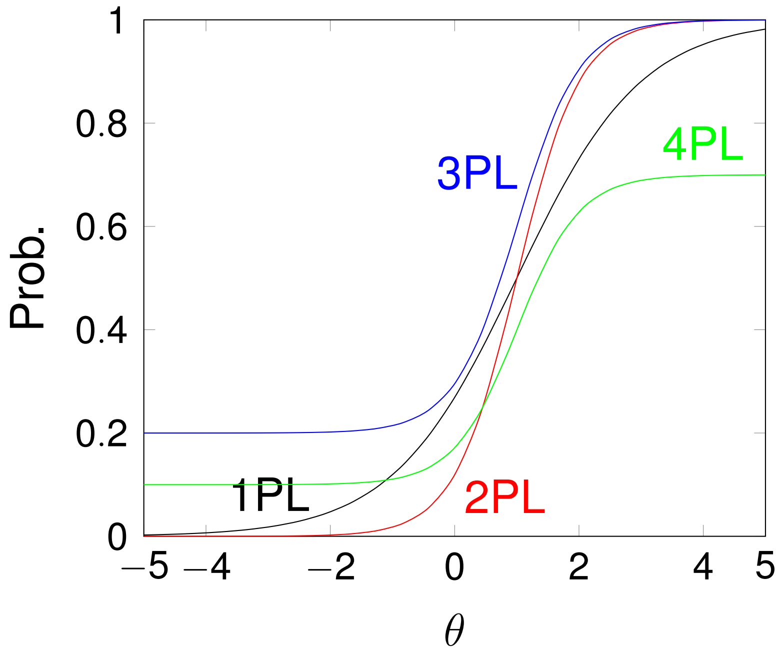
# A sequential hypothesis

By borrowing a terminology used by Hutchinson (1991), it appears that many models can be systematized by considering two sequential 'processes', an ability-based one (*p*-process) and a random-chance-based one (*g*-process).

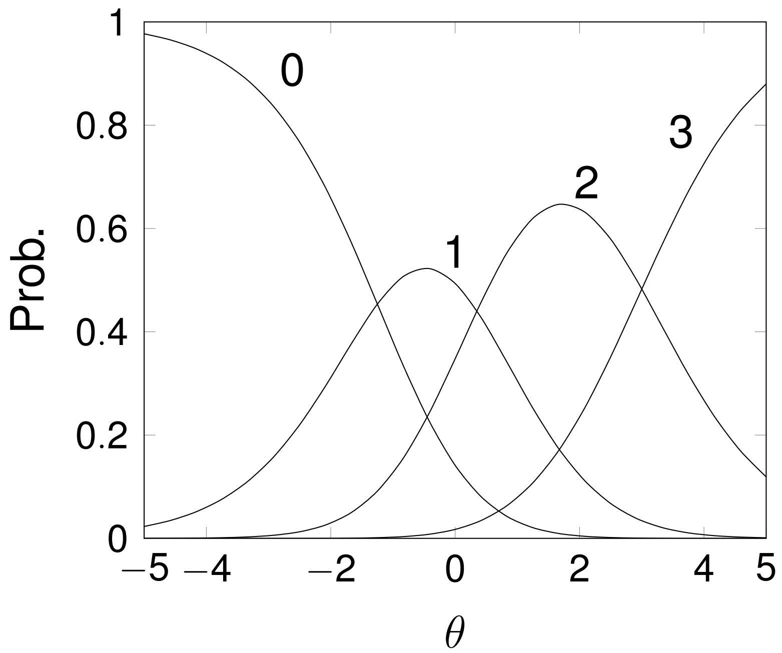


# Item Response Theory

**Dichotomous items:**
e.g., $n$-Parameter Logistic models




**Polytomous items:**
e.g., Partial Credit Model




$$P(X_i = 1|\theta) = c_i + (1 - c_i - d_i)\frac{\exp(a_i(\theta - b_i))}{1 + \exp(a_i(\theta - b_i))}$$

$$P(X_i = k_i|\theta) = \frac{\exp[\sum_{j=0}^{k_i}(\theta - b_{ij})]}{\sum_{h=0}^{3}\exp[\sum_{j=0}^{h}(\theta - b_{ij})]}$$

Local Independence is typically assumed: $P(X = x|\theta) = \prod_i P(X_i = x_i|\theta)$

# Cognitive Diagnostic Models/Assessment

- Whereas IRT considers continuous latent traits, CDMs profile individuals by latent classes $C \in 2^S$, with $S$ a set of dichotomous skills/attributes
- The relation between items and skills is captured by a **Q**-matrix. For instance, given $Q = \{a, b, c\}$ and $S = \{s_1, s_2, s_3\}$

| **Q** | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $a$ | 1 | | |
| $b$ | | 1 | |
| $c$ | 1 | | 1 |

- The **Core CDM** is a latent class model of the form

$$P(X = x) = \sum_{C \in 2^S} \nu(C) P(X = x | C, \mathbf{Q}) = \sum_{C \in 2^S} \nu(C) \prod_i P(X_i = x_i | C, \mathbf{Q}_i)$$

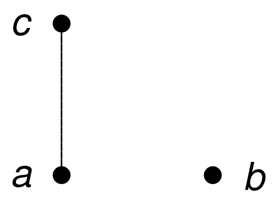
  where $\nu(C)$ is the membership of the latent class $C$ and $P(X_i = x_i | C, \mathbf{Q}_i)$ is the probability of a response $X_i = x_i$ to item $i$ by an individual in class $C$
- Different models assume different forms of $P(X_i = x_i | C, \mathbf{Q}_i)$
- General frameworks apply link functions $\ell[P(X_i = x_i | C, \mathbf{Q}_i)] = f(X_i, C, \mathbf{Q}_i)$

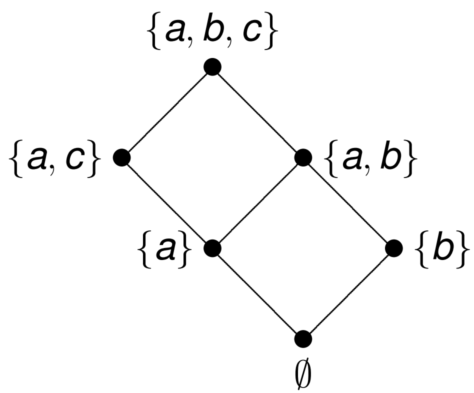
# Knowledge Structure Theory

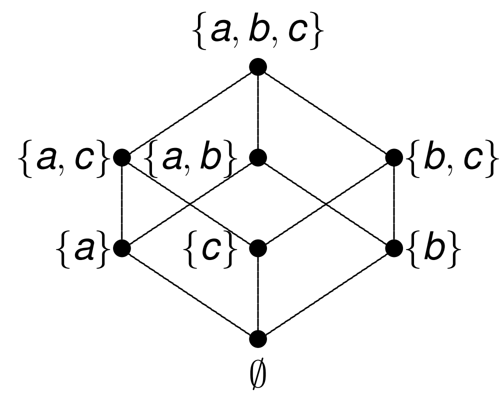
KST profiles individuals by the collections of items they are capable to master. Dependences between items in a domain $Q$ define a probabilistic structure $(Q, \mathcal{K}, \pi)$ with $\{\emptyset, Q\} \subseteq \mathcal{K} \subseteq 2^Q$ and $\pi(K)$ a probability for $K \in \mathcal{K}$.

Structure: $(Q, \mathcal{K}, \pi)$ Responses: $(Q, 2^Q, P)$

Domain: $Q = \{a, b, c\}$




The probability of a pattern of responses $X \subseteq Q$ is given by the BLIM

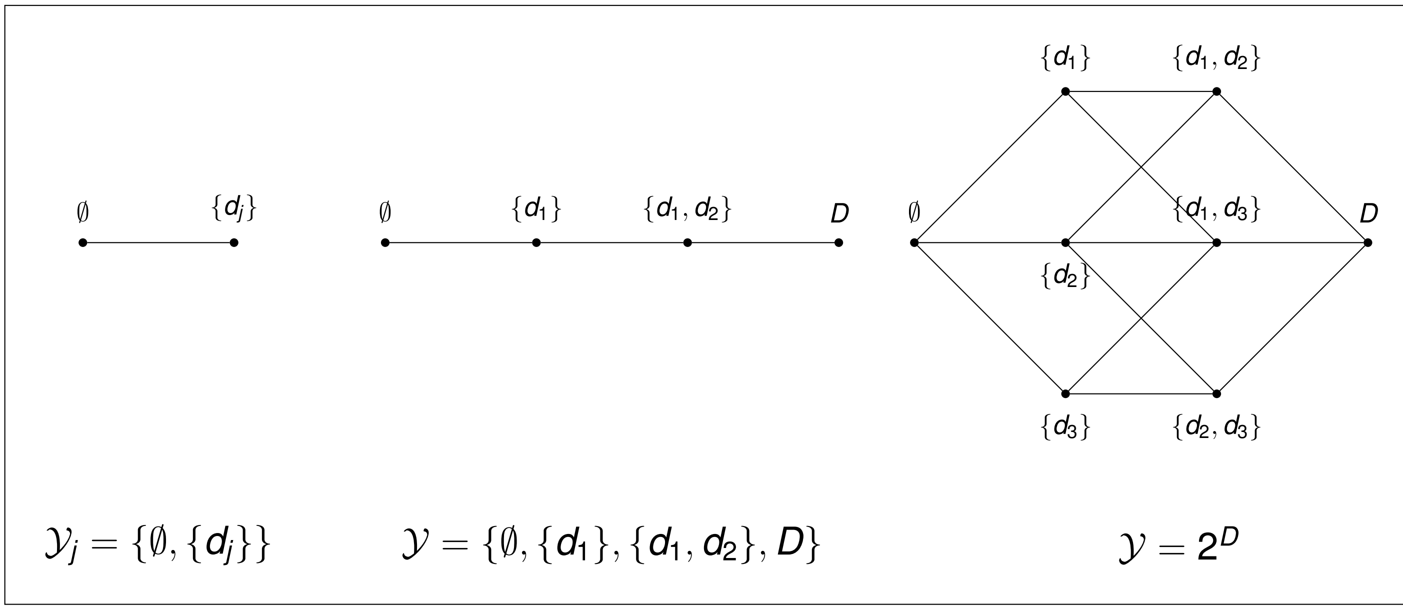
$$P(X) = \sum_{K \in \mathcal{K}} P(X|K)\pi(K) \quad \text{with} \quad P(X|K) = \prod_{q \in K \setminus X} \beta_q \prod_{q \in K \cap X} (1 - \beta_q) \prod_{q \in X \setminus K} \eta_q \prod_{q \in \overline{K} \cap \overline{X}} (1 - \eta_q)$$

where $\eta_q$ and $\beta_q$ are lucky guess and careless error probabilities on item $q$

# Primitive I: Structure

- A **probabilistic structure** is a triple $(D, \mathcal{Y}, P_{\mathcal{Y}})$ with
  - $D$ a **domain** of atomic elements $d_j$
  - $\mathcal{Y}$ a **family** of subsets $Y \subseteq D$, called **states**, such that $\{\emptyset, D\} \subseteq \mathcal{Y} \subseteq 2^D$
  - $P_{\mathcal{Y}}$ a **probability distribution** over $\mathcal{Y}$




- Presently, the atomic elements have no interpretation/semantic, they could be anything (e.g., items, skills, responses, thresholds)

# Primitive I: Structure

- Set-theoretic notation can be converted into vector notation

| KST notation | | CDA/IRT notation | |
|---|---|---|---|
| **Name** | **Symbol** | **Name** | **Symbol** |
| State variable | $Y \in \mathcal{Y}$ | Random vector | $Y : \mathcal{Y} \to \{0,1\}^{\lvert D \rvert}$ |
| State (latent class) | $Y = \{d_j\}_{j \in J}$ | Binary vector | $Y(\{d_j\}_{j \in J}) = y \in \{0,1\}^{\lvert D \rvert}$ |
| Indicator function | $\mathbf{1}_Y(d_j)$ | Random variable | $Y_j : \mathcal{Y} \to \{0,1\}$ |
| Atomic element | $d_j \in Y \subseteq D$ | Realization | $y_j \in \{0,1\}$ |
| Probability measure | $P_{\mathcal{Y}} : \mathcal{Y} \to [0,1]$ | Probability measure | $P_{\mathcal{Y}} : \{0,1\}^{\lvert D \rvert} \to [0,1]$ |
| Probability vector | $P_{\mathcal{Y}} = [P(Y)]^{\mathsf{T}}_{Y \in \mathcal{Y}}$ | Probability vector | $P_{\mathcal{Y}} = [P(Y = y)]^{\mathsf{T}}_{y \in \{0,1\}^{\lvert D \rvert}}$ |

- From the perspective of latent variable models the states $Y \in \mathcal{Y}$ can be seen as ordered latent classes
- Nominal latent classes are collections of singletons, i.e.

$$\mathcal{Y} = \{\{d_1\}, \{d_2\}, \ldots, \{d_{\lvert D \rvert}\}\}$$

with every singleton corresponding to a latent class.

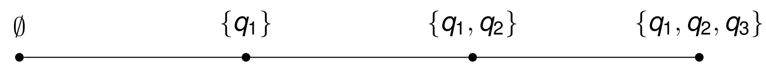
 

# Primitive I: Structure

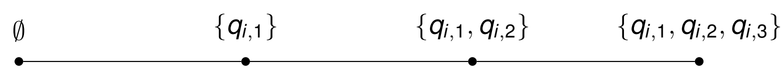
- Different **sorts** of structures follow from different interpretations of the atomic elements in the domain

$\mathcal{K}$ **Knowledge structures** $(Q, \mathcal{K}, \pi_{\mathcal{K}})$
Structures that consider items $q_i$ or sub-items/item thresholds $q_{i,n}$

Chain of items: $\mathcal{K} = \{\emptyset, \{q_1\}, \{q_1, q_2\}, \{q_1, q_2, q_3\}\}$




Polytomous item: $\underline{\mathcal{K}}_i = \{\emptyset, \{q_{i,1}\}, \{q_{i,1}, q_{i,2}\}, \{q_{i,1}, q_{i,2}, q_{i,3}\}\}$



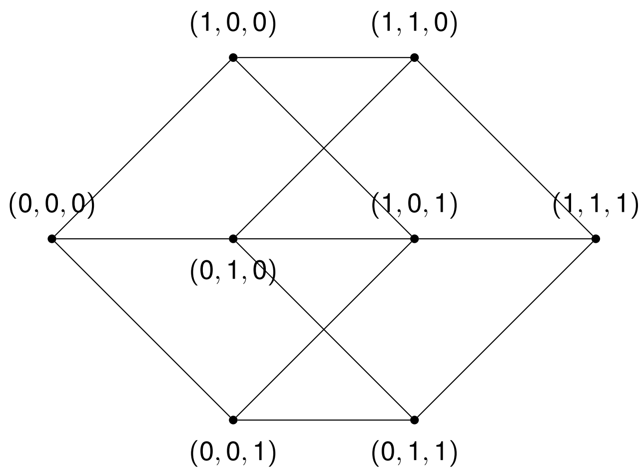

IRT models for dichotomous items correspond to the power set $\mathcal{K} = 2^Q$ case, while IRT models for polytomous items correspond to the chain

# Primitive I: Structure

$\mathcal{X}$ **Response structures** $(R, \mathcal{X}, P_{\mathcal{X}})$

$$\text{Power set:} \quad \mathcal{X} = 2^R$$
$$\text{Chain:} \quad \mathcal{X} = \{\emptyset, \{r_1\}, \{r_1, r_2\}, \{r_1, r_2, r_3\}\}$$

KST, CDA, and IRT models for dichotomous items assume $R = Q$ and $\mathcal{X} = 2^Q$, that is , using directly the random vector notation for $X \subseteq Q$



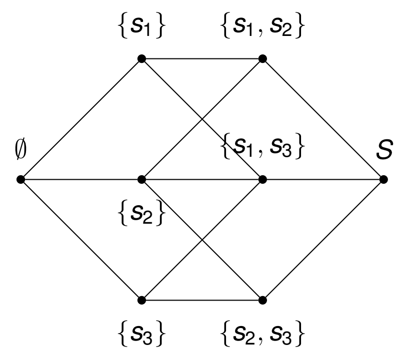

IRT models for polytomous items assume instead a chain

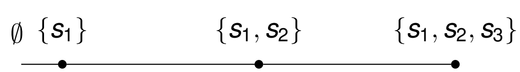
# Primitive I: Structure

$\mathcal{C}$ **Competence structures** $(S, \mathcal{C}, \nu_{\mathcal{C}})$
Structures of attributes $s_a$, or sub-attributes/attribute thresholds $s_{a,n}$

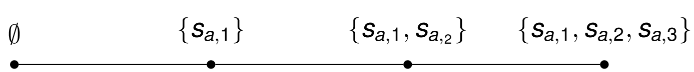
Power set of attributes: $\mathcal{C} = 2^S$




Chain of attributes: $\mathcal{C} = \{\emptyset, \{s_1\}, \{s_1, s_2\}, \{s_1, s_2, s_3\}\}$




Polytomous attribute: $\underline{\mathcal{C}}_a = \{\emptyset, \{s_{a,1}\}, \{s_{a,1}, s_{a,2}\}, \{s_{a,1}, s_{a,2}, s_{a,3}\}\}$



# Primitive I: Structure

- CDMs typically assume a power set of attributes while IRT models require to ‘extend’ a chain of attribute thresholds to an ‘increasingly rich’ attribute with an ‘increasingly’ rich domain of thresholds
- An *infinite competence structure* is a pair $(\underline{S}_a, \underline{\mathcal{C}}_a)$ consisting of
  a) a domain of thresholds $\underline{S}_a = \{s_{a,n}\}_{n \in \mathbb{I}^+}$ with index sets $\mathbb{I} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$
  b) a family $\underline{\mathcal{C}}_a$ in which every non-empty competence state collects all thresholds that precede a given one, that is

$$\underline{\mathcal{C}}_a = \{C(t)\}_{t \in \overline{\mathbb{I}}^{\geq}} \quad \text{with} \quad \begin{cases} C(0) = \emptyset \\ C(t) = \{s_{a,n} \in \underline{S}_a : n \leq t, n \in \mathbb{I}^+\} \\ C(\infty) = \underline{S}_a \end{cases}$$

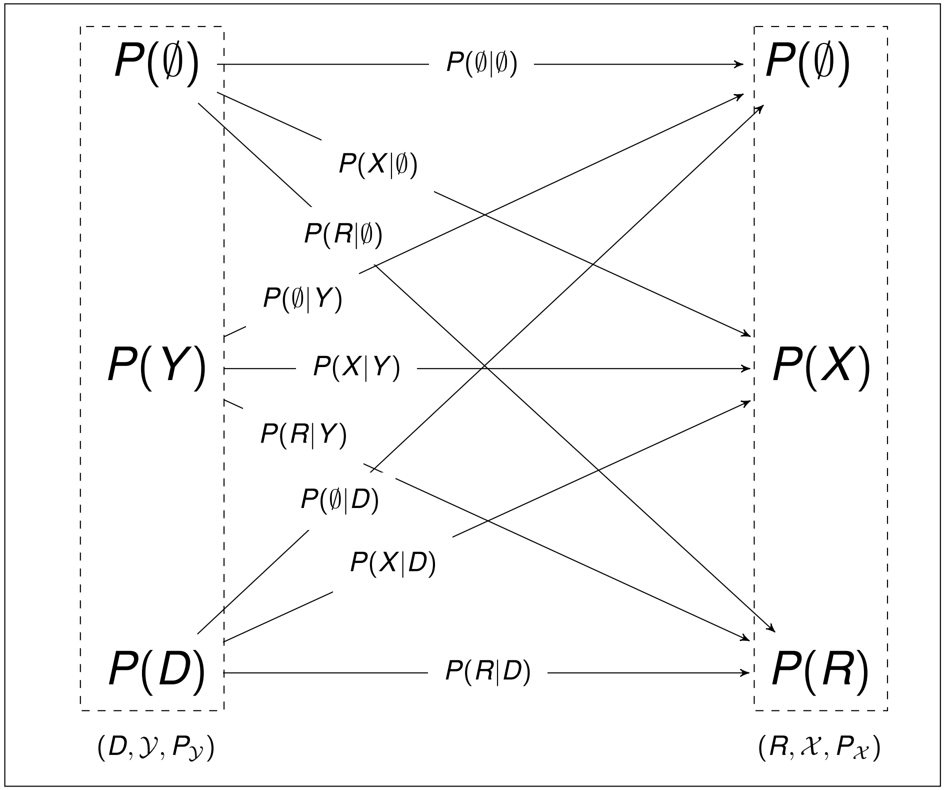
  and non-negative $\mathbb{I}^{\geq}$, positive $\mathbb{I}^+$, and extended $\overline{\mathbb{I}} = \mathbb{I} \cup \{-\infty, +\infty\}$ indices.
- **Latent traits** are defined as (realizations of) random variables defined over infinite competence structures

$$\theta : \underline{\mathcal{C}}_a \to \mathbb{R}$$

# Primitive II: Process

- Given two structures $(D, \mathcal{Y}, P_\mathcal{Y})$ and $(R, \mathcal{X}, P_\mathcal{X})$, a **process** is a pair $(\mathcal{Y}, \mathcal{X})$ such that it holds $P_\mathcal{X} = P_{\mathcal{X}|\mathcal{Y}} \cdot P_\mathcal{Y}$, i.e. element-wise for all $X \in \mathcal{X}$

$$P(X) = \sum_{Y \in \mathcal{Y}} P(X|Y)P(Y)$$



# Primitive II: Process

- A **stochastic** process generalizes Latent Variable Models (LVMs)
- If $\mathcal{Y}$ is a collection of latent classes $Y$, one has latent class models

$$P(X) = \sum_{Y \in \mathcal{Y}} P(X|Y)P(Y)$$

e.g., $(2^S, 2^Q)$ is the core CDM and $(\mathcal{K}, 2^Q)$ the basic KST model

$$P(X) = \sum_{C \in 2^S} P(X|C)\nu(C), \quad P(X) = \sum_{K \in \mathcal{K}} P(X|C)\pi(K).$$

- If $\mathcal{Y}$ is an infinite competence structure, one has latent trait models

$$P(X) = \int_{Y \in \mathcal{Y}} P(X|Y)dP(Y)$$

e.g., $(\underline{\mathcal{C}}, 2^Q)$ is an IRT model

$$P(X) = \int_{C \in \underline{\mathcal{C}}} P(X|C)dP(C) = \int P(X|\theta)\nu(\theta)d\theta$$

for $\theta : \underline{\mathcal{C}} \to \mathbb{R}$ with pdf $\nu(\theta)$ associated to the measure $P_{\mathcal{Y}}$

# Primitive II: Process

- But a relation between structures can also be deterministic in nature
- A **deterministic** process returns deterministic relations which are typical of KST and CDA/CDMs

KST A process $(\mathcal{C}, \mathcal{K})$ is a **problem function** $p : 2^S \to 2^Q$ since it holds

$$P(K|C) = \xi_{KC} = \mathbf{1}_{p^{-1}(\{K\})}(C) = \begin{cases} 1 & \text{p(C)=K} \\ 0 & \text{otherwise} \end{cases}$$

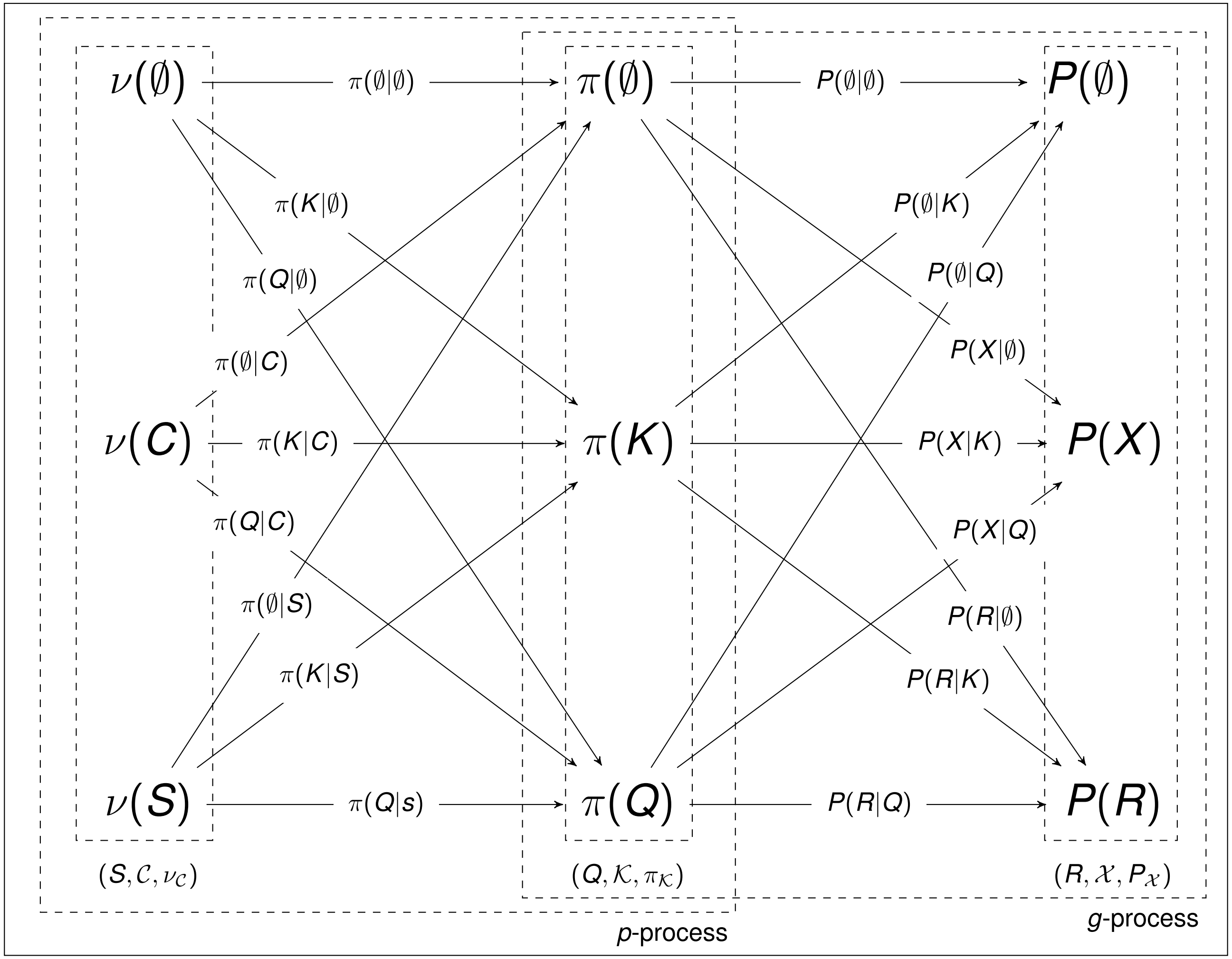
for $K \in \mathcal{K}$ and $C \in \mathcal{C}$

CDA A process $(\mathcal{C}, \mathcal{K})$ is the most general of three types of **Q**-matrices

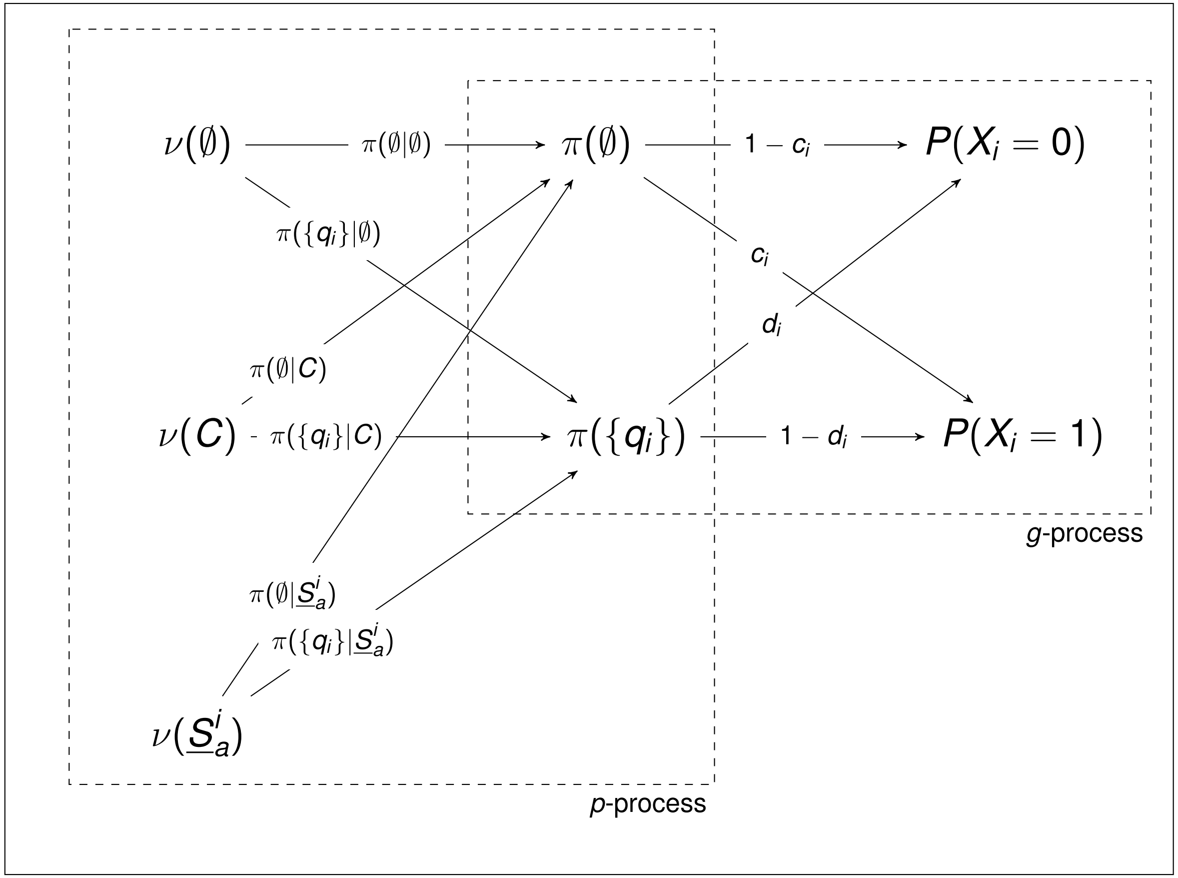
| Q-matrix | Symbol | Elements | Links | To |
|---|---|---|---|---|
| Generalized | $\mathbf{Q}_{\mathcal{K},\mathcal{C}}$ | $\xi_{KC} \in \{0, 1\}$ | knowledge state $K \in \mathcal{K}$ | competence state $C \in \mathcal{C}$ |
| Extended | $\mathbf{Q}^*$ | $\xi_{iC} \in \{0, 1\}$ | item $q_i \in Q$ | competence state $C \in \mathcal{C}$ |
| Traditional | $\mathbf{Q}$ | $\xi_{ia} \in \{0, 1\}$ | item $q_i \in Q$ | skill $s_a \in S$ |

# Sequential hypothesis with two-processes

Many models can be systematized by considering two processes sequentially.



# Basic structure of a uni-dimensional 4-parameter IRT model




When the competence structure is infinite, the processes yield a 4-parameter IRT model

$$P(X_i = 1|\theta) = c_i + (1 - c_i - d_i)\pi(\{q_i\}|\theta)$$

# Structures and Processes combined yield model 'Templates'

- Structures and processes can be combined in different ways

i) **sequentially** (memoryless)
Given two processes $(\mathcal{C}, \mathcal{K})$ and $(\mathcal{K}, \mathcal{X})$, a memoryless two-processes model is given by a triple $(\mathcal{C}, \mathcal{K}, \mathcal{X})$ such that for all $X \in \mathcal{X}$ it holds

$$P(X) = \sum_{C \in \mathcal{C}} \sum_{K \in \mathcal{K}} P(X|K)\pi(K|C)\nu(C).$$

or in vector notation $P_{\mathcal{X}} = P_{\mathcal{X}|\mathcal{K}} \cdot \pi_{\mathcal{K}|\mathcal{C}} \cdot \nu_{\mathcal{C}}$

ii) **in parallel** (non-memoryless) and dependent
A process of the form $(\mathcal{C} \times \mathcal{K}, \mathcal{X})$ such that

$$P(X) = \sum_{C \in \mathcal{C}} \sum_{K \in \mathcal{K}} P(X|C, K)\pi(K|C)\nu(C)$$

iii) **in parallel** (non-memoryless) but independent
A process of the form $(\underline{\mathcal{C}}_1 \times \underline{\mathcal{C}}_2, \mathcal{X})$ such that

$$P(X) = \sum_{C \in \underline{\mathcal{C}}_1} \sum_{C \in \underline{\mathcal{C}}_2} P(X|C_1)P(X|C_2)\nu(C_1, C_2).$$

iv) ...

# Templates can be given different interpretations

Competence, knowledge, and response structures of different sorts can be specified in the processes thus providing different interpretations

| Triple | Models for |
|---|---|
| $(\mathcal{C}, \mathcal{K}, \mathcal{X})$ | dichotomous items and attributes |
| $(\underline{\mathcal{C}}_a, \mathcal{K}, \mathcal{X})$ | dichotomous items depending on a polytomous attribute (uni-dimensional) |
| $(\mathcal{C}, \mathcal{K}_i, \mathcal{X}_i)$ | a dichotomous item depending on dichotomous attributes |
| $(\underline{\mathcal{C}}_a, \mathcal{K}_i, \mathcal{X}_i)$ | a dichotomous item depending on a polytomous attribute (uni-dimensional) |
| $(\underline{\mathcal{C}}_a \times \underline{\mathcal{C}}_{a'}, \mathcal{K}_i, \mathcal{X}_i)$ | a dichotomous item depending on two polytomous attributes (multi-dimensional) |
| $(\mathcal{C}, \underline{\mathcal{K}}_i, \mathcal{X}_i)$ | a polytomous item depending on dichotomous attributes |
| $(\underline{\mathcal{C}}_a, \underline{\mathcal{K}}_i, \mathcal{X}_i)$ | a polytomous item and attribute (trait) |
| $(\underline{\mathcal{C}}_a \times \underline{\mathcal{C}}_{a'}, \underline{\mathcal{K}}_i, \mathcal{X}_i)$ | a polytomous item depending on two polytomous attributes (multi-dimensional) |
| ... | ... |

# 2 Operations on processes

- The conditional probabilities $P_{\mathcal{X}|\mathcal{Y}}$ are subject to two main operations

i) **Factorization**

Some form of probabilistic or conditional independence is assumed.
E.g., (strong) local stochastic independence (LI)

$$P(X|Y) = \prod_{r_k \in R} P(X_k = 1|Y)^{X_k}(1 - P(X_k = 1|Y))^{1-X_k}$$

Generalized Local Independence (GLI, Noventa et al. 2019)

$$P(X|Y) = \prod_{r_k \in X} P(X_k = 1|Y) \prod_{r_k \in X^{\mathcal{O}}} (1 - P(X_k = 1|Y))$$

where $X^{\mathcal{O}}$ is the outer fringe of $X$

$$X^{\mathcal{O}} = \{r \in R | X \cup \{r\} \in \mathcal{X}\}$$

- For $\mathcal{X} = 2^R$, GLI returns LI

# 2 Operations on processes

ii) **Reparametrization**

conditional probabilities are transformed via link functions as in generalized latent variable models (GLVMs)

Link functions (logit, probit, log, identity) are set

$$\ell_r[P(X|Y)] = f_r(X, Y) \quad \text{for some} \quad f_r : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$$
$$\ell_{r_k}[P(X_k|Y)] = f_{r_k}(X_k, Y) \quad \text{for all} \quad Y \in \mathcal{Y} \quad \text{and some} \quad f_{r_k} : \mathcal{X}_k \times \mathcal{Y} \to \mathbb{R}$$

a linear kernel is typically selected

$$f_{r_k}(X_k = x_k, Y) = \sum_{T \in \mathcal{Y}, T \subseteq Y} \lambda_{x_k, T}$$

and can be shown to yield

a) CDA kernel for finite competence structures

b) IRT kernel for infinite competence structures

Introduction

2 Primitives

2 Operations

# Results: Taxonomy

Conclusions

Appendix

# Taxonomy

Different KST, CDA, and IRT models can then be derived depending on

1. Which **structures** are chosen?
2. How many **processes** are involved to combine structures?
   i) 1 process
   ii) 2 memoryless processes
   iii) 2 non-memoryless processes
   iv) ...
3. How conditional probabilities in the processes are set?

   (There are some common assumptions in literature)

3bis. Is some form of independence (**factorization**) assumed on ... ?
   i) items
   ii) attributes
   iii) thresholds
   iv) responses
   v) ...

3ter. Which link function (identity, logit, probit, logarithm) is chosen and which terms in the kernel are retained (**reparametrization**)?
   i) positive $\lambda$s - conjunctive models
   ii) some negative $\lambda$ - disjunctive models
   iii) only $\lambda$ for main effects - main effects models

# Common assumptions on the *p*- and *g*-processes

- The *g*-process if present is
  1. always GLI-factorized
  2. either assumed to be competence independent $P(X|K)$ or dependent $P(X|K, C)$
- The *p*-process is
  1. sometimes GLI-factorized but not always
  2. sometimes assumed to be **independent** on competence

$$\pi_{\mathcal{K}|\mathcal{C}} = \pi_{\mathcal{K}}$$

  3. sometimes assumed to be **deterministic**

$$\pi_{\mathcal{K}|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}}$$

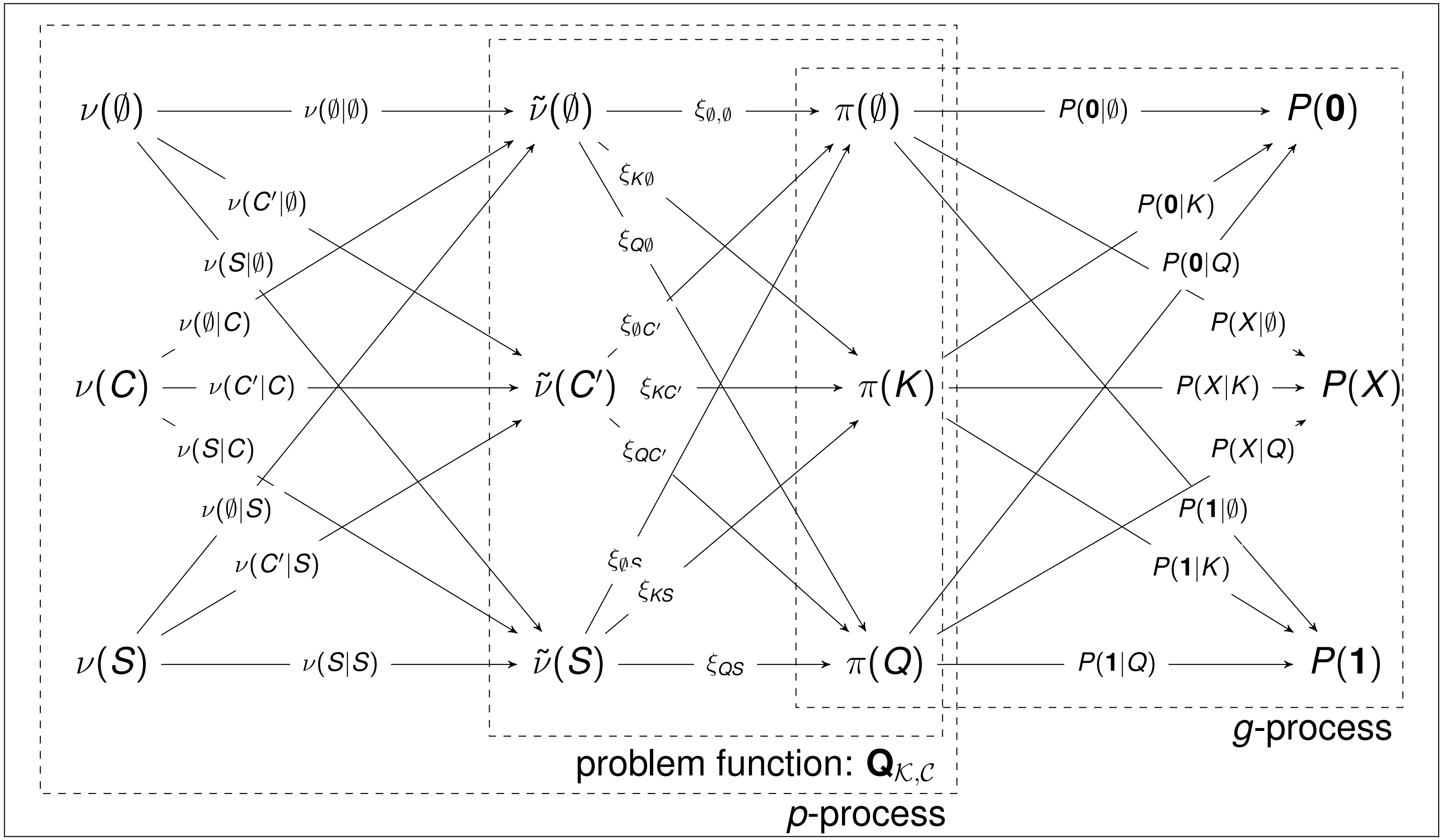
  4. either assumed to allow or not independence of the attributes and their processes
  5. sometimes assumed to allow for ‘changes’ in competence
     **Effective competence state**:

$$\pi_{\mathcal{K}|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}} \cdot \nu_{\mathcal{C}|\mathcal{C}}$$

Responses are obtained as if the individual was in a different competence state. This is considered by an additional process $(\mathcal{C}, \mathcal{C})$ and allows to rewrite $\boldsymbol{\pi}_{K|C}$ as

$$\pi(K|C) = \sum_{C' \in p^{-1}(K)} \nu(C'|C).$$

# Basic structure of KST and CDA models



# Taxonomy: pure *p*-process, GLVMs

| *g*-process | *p*-process | Independent<br>$\pi_{\mathcal{K}\mid\mathcal{C}} = \pi_{\mathcal{K}}$ | Deterministic<br>$\pi_{\mathcal{K}\mid\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}}$ | No independence on attributes | Effective competence state and independent attributes | Parallel and Independent processes on attributes |
|---|---|---|---|---|---|---|
| No *g*-process<br>No left-side added structure | No GLI<br>on $\pi_{\mathcal{K}\mid\mathcal{C}}$ | Simple proportions<br>and log-linear models | Proportions based<br>on Problem function | **KST**: State Response Function<br>**CDA** Main effects models:<br>NIDO, R-RUM,<br>C-RUM/LLM, ACDM<br>**IRT-NIRT** Single-item IRFs:<br>a) Unidimensional:<br>e.g., 1-2PL (Rasch, Birnbaum),<br>1-2PNO, ARRG, LLTM,<br>Q-Diffusion, MHM, DMM<br>b) Polytomous Divide-by-Total:<br>e.g., BNM, RSM, PCM,<br>GPCM, GPMNM, GRM[2]<br>c) Polytomous Difference models:<br>e.g., GRM[2]<br>**MIRT** Single-item IRFs:<br>a) Multidimensional:<br>e.g., M2PL, MNO<br>b) Polytomous Divide-by-Total:<br>e.g., MPCM, MGPCM<br>c) Polytomous difference models:<br>e.g., MGRM | **CDA** Single-item IRFs:<br>a) Conjunctive models:<br>NIDA, MCLCM-C,<br>(Full) NC-RUM;<br>b) Disjunctive models:<br>MCLMC-D | **MIRT** Single-item IRFs:<br>e.g., MLTM-D |
| | GLI<br>on $\pi_{\mathcal{K}\mid\mathcal{C}}$ | **KST**: SLM<br>Simple proportions and<br>independence on items.<br>Independence and<br>quasi-independence<br>in log-linear models | Proportions based<br>on Problem function<br>and Independence<br>on items | **KST**: Θ-SLM;<br>**IRT-MIRT**: a) Local independence<br>on multiple items using the<br>IRFs given above<br>b) Sequential/step models,<br>GRM[1], Acceleration, CRMs; | **CDA**:<br>Local Independence<br>on multiple items using<br>the IRFs given above | **MIRT**:<br>Local Independence<br>on multiple items using<br>the IRFs given above |
| | **General CDA frameworks**: GDINA, LCDM, GDM | | | | | |

# Taxonomy: $p + g$-process, left-side added GLVMs

| $g$-process | $p$-process | Independent $\pi_{\mathcal{K}\|\mathcal{C}} = \pi_{\mathcal{K}}$ | Deterministic $\pi_{\mathcal{K}\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}}$ | No independence on attributes | Effective competence state and independent attributes | Parallel and Independent processes on attributes |
|---|---|---|---|---|---|---|
| GLI on $g$-process Left-side added structure | No GLI on $\pi_{\mathcal{K}\|\mathcal{C}}$ | **KST**: BLIM Log-linear model formulation of LCA | **KST**: CBLIM **CDA**: MS-DINA DINA, DINO RDINA, RDINO HO-DINA | **KST**: Θ-BLIM, LKS **IRT-MIRT** Single-item IRFs: a) Unidimensional: e.g., 3-4PL, 3-4PNO, Choppin's extension of 3PL b) Multidimensional: e.g., M3PL, M3NO c) Polytomous: e.g., Samejima's extension of BNM | **CDA** Single-item IRFs Unified Model, HO-PINC, HO-DINA | **CDA**: PINC **MIRT** Single-item IRFs: e.g., MLTM, GLTM |
| | GLI on $\pi_{\mathcal{K}\|\mathcal{C}}$ | **KST**: BLIM+SLM | **KST-CDA**: Local independence on multiple items using the IRFs given above | **KST**: Θ-BLIM+Θ-SLM **IRT-MIRT**: Local independence on multiple items using the IRFs given above | **CDA**: Local independence on multiple items using the IRFs given above | **MIRT**: Local independence on multiple items using the IRFs given above |
| | **General CDA frameworks**: GDINA, LCDM, GDM | | | | | |
| GLI on Competence-based $g$-process and left-side added structure | No GLI on $\pi_{\mathcal{K}\|\mathcal{C}}$ | **MM-IRT**: MRM, Hybrid[5], Discrete Mixture Models | Generalized versions of KST and CDA models with competence-based error parameters | **IRT-MIRT**: Single-item IRFs a) Unidimensional: e.g., 1PL-AG, 2PLE b) Multidimensional: e.g., M2PL-AIG | Generalized versions of CDA models with competence-based error parameters | Generalized versions of MIRT models with competence-based error parameters |
| | GLI on $\pi_{\mathcal{K}\|\mathcal{C}}$ | Combination of MM-IRT with KST state factorization of membership probability | Generalized versions of KST and CDA models with competence-based error parameters | **IRT-MIRT**: Local independence on multiple items using the IRFs given above | Generalized versions of CDA models with GLI and competence-based error parameters | Generalized versions of MIRT models with GLI and competence-based error parameters |
| | **General CDA frameworks**: MGDM | | | | | |

# Conclusions

1. Two main primitives: ‘structure’ and ‘process’
2. Two main operations: ‘factorization’ and ‘reparametrization’
3. A two-processes approach provides a practical taxonomy
4. KST and ‘early CDA’ models appear to be based on the same theory applied to atomic entities of different sorts
5. ‘Modern CDA’ approaches appear to be discrete versions of IRT models (i.e., ‘one kernel to derive them all’)
6. General CDA frameworks/models correspond in essence to choices of link functions and kernels
7. Latent-trait extended versions of KST models generalize IRT models (e.g., they allow for alternative ways of modelling local dependence)

# Thanks for your attention!

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# Primitive I: Structures

| Structure | Elements - Random variables | | Domain | State | Structure | Probability |
|---|---|---|---|---|---|---|
| **Knowledge** | | | | | | |
| Dichotomous items | *i*-th item<br>(Std) realization | $q_i$<br>$K_i = k_i$ | $Q$<br>$\{0, 1\}$ | $K$ | $\mathcal{K}$ | $\pi_{\mathcal{K}} = [\pi(K)]^{\mathsf{T}}_{K \in \mathcal{K}}$ |
| Polytomous Item | *n*-th threshold<br>item realization<br>(Std) realization<br>threshold realization | $q_{i,n}$<br><br>$K_i = k_i$<br>$K_{i,n} = k_{i,n}$ | $\underline{Q}_i$<br>$\{\kappa_{i,t}\}_{t \in \{0,\ldots,m_i\}}$<br>$\{0, \ldots, m_i\}$<br>$\{0,1\}$ | $K$ | $\underline{\mathcal{K}}_i$ | $\pi_{\underline{\mathcal{K}}_i} = [\pi(K)]^{\mathsf{T}}_{K \in \underline{\mathcal{K}}_i}$ |
| **Competence** | | | | | | |
| Dichotomous Attributes | *a*-th attribute<br>(Std) realization | $s_a$<br>$C_a = \alpha_a$ | $S$<br>$\{0, 1\}$ | $C$ | $\mathcal{C}$ | $\nu_{\mathcal{C}} = [\nu(C)]^{\mathsf{T}}_{C \in \mathcal{C}}$ |
| Categorical Attribute | *n*-th threshold<br>attribute realization<br>(Std) realization<br>threshold realization | $s_{a,n}$<br>$C_a = c_a$<br>$C_a = \alpha_a$<br>$C_{a,n} = c_{a,n}$ | $\underline{S}_a$<br>$\{\alpha_{a,t}\}_{t \in \{0,\ldots,m_a\}}$<br>$\{0, \ldots, m_a\}$<br>$\{0,1\}$ | $C$ | $\underline{\mathcal{C}}_a$ | $\nu_{\underline{\mathcal{C}}_a} = [\nu(C)]^{\mathsf{T}}_{C \in \underline{\mathcal{C}}_a}$ |
| **Response** | | | | | | |
| All forms of Responses | *k*-th response<br>(Std) realization | $r_k$<br>$X_k = x_k$ | $R$<br>$\{0, \ldots, m_k\}$ | $X$ | $\mathcal{X}$ | $P_{\mathcal{X}} = [P(X)]^{\mathsf{T}}_{X \in \mathcal{X}}$ |

# Taxonomy: *p*-process, GLVMs

| *g*-process | *p*-process | Independent $\pi_{\mathcal{K}\mid\mathcal{C}} = \pi_{\mathcal{K}}$ | Deterministic $\pi_{\mathcal{K}\mid\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}}$ | No independence on attributes | Effective competence state and independent attributes | Parallel and Independent processes on attributes |
|---|---|---|---|---|---|---|
| No *g*-process, $P_{\mathcal{X}\mid\mathcal{K}} = I_{\mid\mathcal{X}\mid}$ $(\mathcal{K} = \mathcal{X})$<br>No left side added structure | No GLI on items on $\pi_{\mathcal{K}\mid\mathcal{C}}$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}}$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}}$<br>$\pi_{\mathcal{K}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}} \cdot \nu_{\mathcal{C}}$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}}$<br>$\pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$\pi_{\mathcal{K}\mid\mathcal{C}} = \ell_q^{-1}[f_q(\mathcal{K},\mathcal{C})]$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$\pi_{\mathcal{K}\mid\mathcal{C}} = \ell_q^{-1}[f_q(\mathcal{K},\mathcal{C})]$<br>$\pi_{\mathcal{K}\mid\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}} \cdot \nu_{\mathcal{C}\mid C}$<br>$\nu_{\mathcal{C}\mid C} = \odot_a \nu_{\mathcal{C}_a\mid C}$<br>$\nu_{\mathcal{C}_a\mid C} = \ell_{s_a}^{-1}[\tilde{\nu}(\mathcal{C}_a, C)]$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$\pi_{\mathcal{K}\mid\mathcal{C}} = \bigotimes_a \pi_{\mathcal{K}\mid\mathcal{C}_a}$<br>$\pi_{\mathcal{K}\mid\mathcal{C}_a} = \ell_q^{-1}[f_q(\mathcal{K},\mathcal{C}_a)]$ |
| | GLI on items on $\pi_{\mathcal{K}\mid\mathcal{C}}$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}}$<br>$\pi_{\mathcal{K}} = \odot_i \pi_{\mathcal{K}_i}$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}}$<br>$\pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$\pi_{\mathcal{K}\mid C} = \odot_i \pi_{\mathcal{K}_i\mid C}$<br>$\pi_{\mathcal{K}_i\mid C} = \mathbf{Q}_{\mathcal{K}_i,\mathcal{C}}$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}}$<br>$\pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid C} \cdot \nu_{C}$<br>$\pi_{\mathcal{K}\mid C} = \odot_i \pi_{\mathcal{K}_i\mid C}$<br>$\pi_{\mathcal{K}_i\mid C} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, C)]$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$\pi_{\mathcal{K}\mid C} = \odot_i \pi_{\mathcal{K}_i\mid C}$<br>$\pi_{\mathcal{K}_i\mid C} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, C)]$<br>$\pi_{\mathcal{K}_i\mid C} = \mathbf{Q}_{\mathcal{K}_i,\mathcal{C}^i} \cdot \nu_{\mathcal{C}^i\mid C}$<br>$\nu_{\mathcal{C}^i\mid C} = \odot_a \nu_{\mathcal{C}^i_a\mid C}$<br>$\nu_{\mathcal{C}^i_a\mid C} = \ell_{s_a}^{-1}[\tilde{\nu}(\mathcal{C}^i_a, C)]$ | $P_{\mathcal{X}} = \pi_{\mathcal{K}} = \pi_{\mathcal{K}\mid\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$\pi_{\mathcal{K}\mid C} = \odot_i \pi_{\mathcal{K}_i\mid C}$<br>$\pi_{\mathcal{K}_i\mid C} = \bigotimes_a \pi_{\mathcal{K}_i\mid\mathcal{C}^i_a}$<br>$\pi_{\mathcal{K}_i\mid\mathcal{C}^i_a} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, \mathcal{C}^i_a)]$ |

# Taxonomy: $p+g$-processes, GLVMs with left side added parameters

| $g$-process | $p$-process | Independent $\pi_{\mathcal{K}\|\mathcal{C}} = \pi_{\mathcal{K}}$ | Deterministic $\pi_{\mathcal{K}\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}}$ | No independence on attributes | Effective competence state and independent attributes | Parallel and Independent processes on attributes |
|---|---|---|---|---|---|---|
| GLI on $P_{\mathcal{X}\|\mathcal{K}} \neq I_{\|\mathcal{X}\|}$ $(\mathcal{K} \subseteq \mathcal{X})$<br>Left-side added structure | No GLI on items on $\pi_{\mathcal{K}\|\mathcal{C}}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \mathbf{Q}_{\mathcal{K},\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}_k\|K} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, K)]$<br>$\pi_{\mathcal{K}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}} \cdot \nu_{\mathcal{C}}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}_k\|K} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, K)]$<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \ell_q^{-1}[f_q(\mathcal{K}, \mathcal{C})]$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}\|\mathcal{C}} = \ell_r^{-1}[f_r(\mathcal{X}, \mathcal{C})]$<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}} \cdot \nu_{\mathcal{C}\|\mathcal{C}}$<br>$\nu_{\mathcal{C}\|\mathcal{C}} = \odot_a \nu_{\mathcal{C}_a\|\mathcal{C}}$<br>$\nu_{\mathcal{C}_a\|\mathcal{C}} = \ell_{s_a}^{-1}[\tilde{\nu}(\mathcal{C}_a, C)]$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}_k\|K} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, K)]$<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \bigotimes_a \pi_{\mathcal{K}\|\mathcal{C}_a}$<br>$\pi_{\mathcal{K}\|\mathcal{C}_a} = \ell_q^{-1}[f_q(\mathcal{K}, \mathcal{C}_a)]$ |
| | GLI on items on $\pi_{\mathcal{K}\|\mathcal{C}}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$\pi_{\mathcal{K}} = \odot_i \pi_{\mathcal{K}_i}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}_k\|K} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, K)]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|C} = \mathbf{Q}_{\mathcal{K}_i,\mathcal{C}}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}_k\|K} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, K)]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|C} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, C)]$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$P_{\mathcal{X}_k\|C} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, C)]$<br>$\pi_{\mathcal{K}_i\|C} = \mathbf{Q}_{\mathcal{K}_i,\mathcal{C}^i} \cdot \nu_{\mathcal{C}^i\|C}$<br>$\nu_{\mathcal{C}^i\|C} = \odot_a \nu_{\mathcal{C}^i_a\|C}$<br>$\nu_{\mathcal{C}^i_a\|C} = \ell_{s_a}^{-1}[\tilde{\nu}(\mathcal{C}^i_a, C)]$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{C}} \cdot \nu_{\mathcal{C}}$<br>$P_{\mathcal{X}\|\mathcal{C}} = P_{\mathcal{X}\|\mathcal{K}} \cdot \pi_{\mathcal{K}\|\mathcal{C}}$<br>$P_{\mathcal{X}\|K} = \odot_k P_{\mathcal{X}_k\|K}$<br>$P_{\mathcal{X}_k\|K} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, K)]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|C} = \bigotimes_a \pi_{\mathcal{K}_i\|\mathcal{C}^i_a}$<br>$\pi_{\mathcal{K}_i\|\mathcal{C}^i_a} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, \mathcal{C}^i_a)]$ |

# Taxonomy: $p + g$-processes, GLVMs with competence-based left side added parameters

| $g$-process | $p$-process | Independent<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \pi_{\mathcal{K}}$ | Deterministic<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}}$ | No independence on attributes | Effective competence state and independent attributes | Parallel and Independent processes on attributes |
|---|---|---|---|---|---|---|
| GLI on $P_{\mathcal{X}\|\mathcal{K},\mathcal{C}} \neq I_{\|\mathcal{X}\|}$ $(\mathcal{K} \subseteq \mathcal{X})$<br>Competence-based Left-side added structure | No GLI on items on $\pi_{\mathcal{K}\|\mathcal{C}}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\pi_{\mathcal{K}} \cdot \nu_{\mathcal{C}}^{\mathsf{T}}]$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \mathbf{Q}_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \ell_q^{-1}[f_q(\mathcal{K}, \mathcal{C})]$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}\|M} = \ell_r^{-1}[f_r(\mathcal{X}, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K},\mathcal{C}} \cdot \nu_{\mathcal{C}\|\mathcal{C}}$<br>$\nu_{\mathcal{C}\|C} = \odot_a \nu_{\mathcal{C}_a\|C}$<br>$\nu_{\mathcal{C}_a\|C} = \ell_{s_a}^{-1}[\tilde{\nu}(\mathcal{C}_a, C)]$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|\mathcal{C}} = \bigotimes_a \pi_{\mathcal{K}\|\mathcal{C}_a}$<br>$\pi_{\mathcal{K}\|\mathcal{C}_a} = \ell_q^{-1}[f_q(\mathcal{K}, \mathcal{C}_a)]$<br>$M = (C, K)$ |
| | GLI on items on $\pi_{\mathcal{K}\|\mathcal{C}}$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\pi_{\mathcal{K}} \cdot \nu_{\mathcal{C}}^{\mathsf{T}}]$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{K}} = \odot_i \pi_{\mathcal{K}_i}$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K}_i,\mathcal{C}}$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|C} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, C)]$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|\mathcal{C}} = \mathbf{Q}_{\mathcal{K}_i,\mathcal{C}^i} \cdot \nu_{\mathcal{C}^i\|\mathcal{C}}$<br>$\nu_{\mathcal{C}^i\|C} = \odot_a \nu_{\mathcal{C}_a^i\|C}$<br>$\nu_{\mathcal{C}_a^i\|C} = \ell_{s_a}^{-1}[\tilde{\nu}(\mathcal{C}_a^i, C)]$<br>$M = (C, K)$ | $P_{\mathcal{X}} = P_{\mathcal{X}\|\mathcal{M}} \cdot \pi_{\mathcal{M}}$<br>$P_{\mathcal{X}\|M} = \odot_k P_{\mathcal{X}_k\|M}$<br>$P_{\mathcal{X}_k\|M} = \ell_{r_k}^{-1}[f_{r_k}(\mathcal{X}_k, M)]$<br>$\pi_{\mathcal{M}} = \mathsf{v}[\langle \pi_{\mathcal{K},\mathcal{C}}, \mathbf{1} \cdot \nu_{\mathcal{C}}^{\mathsf{T}} \rangle_H^{\mathsf{T}}]$<br>$\pi_{\mathcal{K}\|C} = \odot_i \pi_{\mathcal{K}_i\|C}$<br>$\pi_{\mathcal{K}_i\|\mathcal{C}} = \bigotimes_a \pi_{\mathcal{K}_i\|\mathcal{C}_a^i}$<br>$\pi_{\mathcal{K}_i\|\mathcal{C}_a^i} = \ell_{q_i}^{-1}[f_{q_i}(\mathcal{K}_i, \mathcal{C}_a^i)]$<br>$M = (C, K)$ |