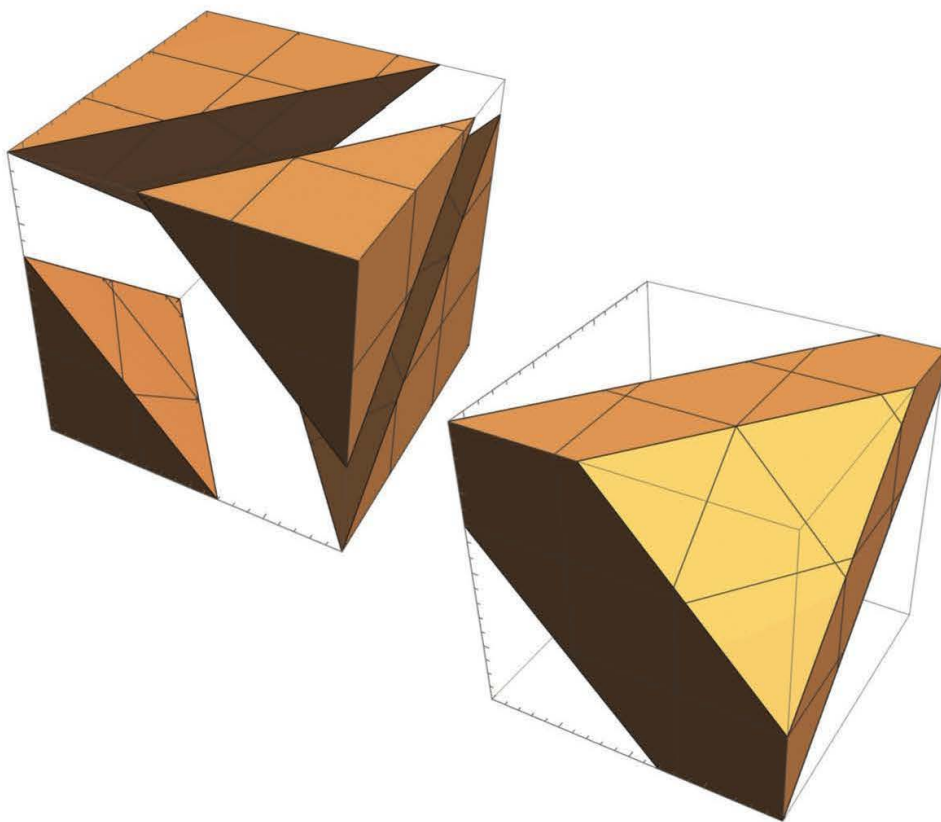


Contextuality in Random Variables

A Systematic Introduction

Ehtibar N. Dzhafarov, Janne V. Kujala,
and Víctor H. Cervantes



Dzhafarov,
Kujala, and
Cervantes

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To Gündüz Sarioğlu, the noble thinker in the desert

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* An asterisk at a section title indicates that the section contains technical material that can be skipped or skimmed on a first reading.

Preface

Subject of the Book

This book is about *systems of random variables*. A system of random variables is a set of random variables ordered in two ways: by their *contents* (the questions the variables answer) and by their *contexts* (conditions under which they are recorded). Such a system can be *contextual* or *noncontextual*. These terms, being derived from the word *context*, have cognates in many areas, but most of them are too syncretic to be helpful. They do not distinguish contextuality from a broader notion of *context-dependence*. The meaning of contextuality in this book is closest to that in quantum physics, where its special forms are known as *nonlocality* and the *Kochen–Specker contextuality*. Another related notion is that of *selectiveness of influences* in cognitive sciences, which is mathematically equivalent to noncontextuality in the Kochen–Specker sense. However, our treatment is more general and more abstract, applying to all systems of random variables, whether they describe phenomena in quantum physics, computer science, or psychology.

The mathematical essence of contextuality is in the *similarity* of random variables answering the same question (having the same content) in different contexts. There is a principled way of measuring how similar two such variables are: by computing the maximal probability with which they could coincide *if* they were jointly distributed (de facto no two variables are if they are in different contexts). A system is contextual if these same-content variables, in order to be compatible with other variables in their respective contexts, have to be more dissimilar than they are when considered in isolation, outside their contexts. If this sounds less than clear, this book will spell it out in great detail.

One fact, however, can be mentioned right away: The difference in the similarity of two random variables considered within and without their contexts

is not due to any physical action of contexts upon the variables. Considerations of causality may be involved in explaining why the variables within this or that context have the joint distributions they are observed or predicted to have. Contextuality, however, neither requires nor admits explanations in terms of physical causes and effects. It is a mathematical property, based on the fact that a random variable is a function whose domain is determined by all other variables it is jointly distributed with. Therefore a variable considered within its context and a variable considered in isolation are simply two different variables, even if they answer the same question. There is no transition between the two, and they properly have to be denoted by different symbols. In fact, contextuality can never be pinned to specific variables, which it would have to if we dealt with causal influences. When a system is contextual, we know that some of the same-content variables are more dissimilar within their contexts than they are without – but one can never say what this difference in similarity is for any particular pair. Whether a system is or is not contextual is a *system-wise* (and *system-wide*) property of the system. In particular, it is easy to construct examples of contextual systems that become noncontextual (or vice versa) following a modification of any single variable's distribution in the system. This applies, in particular, to variables recorded prior to all other variables in their contexts, excluding thereby any possibility of causal influences being involved.

Philosophy of the Book

It is the leitmotif of this book that contextuality-related concepts are purely mathematical constructs. To preclude a hasty reaction to this statement: Something being purely mathematical does not prevent it from being useful and widely applicable. Matrices, integrals, and arithmetic operations are purely mathematical constructs, based on no empirical assumptions and subject to no empirical verification. They nevertheless have numerous applications, and they are highly useful in all of them. It is simply that no physical considerations are involved in determining if a matrix is invertible or if the value of an integral is finite.

Although the interplay between the mathematical and the empirical is a deep philosophical issue which is outside the reach of this book, one can acknowledge some obvious facts. Mathematical analysis (in our case, contextuality analysis) always applies to mathematical descriptions of what is being studied. These descriptions (in our case, systems of random variables) are always non-unique, but they are not arbitrary. A situation they describe constrains them and makes them interrelated. One can say that it is an

empirical property of a situation to afford a certain class of mathematical descriptions and only them. However, mathematical properties of different descriptions in this class (in our case, whether they are contextual or not) may differ. Interpreting contextuality-related concepts in terms of hypothetical natural phenomena may be innocuous if these phenomena are viewed as mere analogies or metaphors. But on a deeper level of analysis such interpretations, as a rule, obscure understanding. Thus, as thoroughly explained in this book, a *hidden variable model* with *free choice* but *context-dependent mapping* of the hidden variables into observable ones is equivalent to a hidden variable model with *context-independent mapping* but compromised freedom of choice. This simple mathematical fact looks paradoxical if not impossible if one interprets hidden variables as physical states or forces, and freedom of choice as a property of a psychological (or machine-generated) process of decision making.

Our approach largely follows the great philosophical traditions of logical positivism and pragmatism. Although they are often eschewed nowadays by scientists and philosophers, their non-radical versions serve as highly useful guides. Logical positivism prevents us from constructing theories based on vague intuitions and connotations of words. Pragmatism complements this by making us realize that competing descriptions are not really different unless they have distinct observable consequences. While these philosophical traditions by no means exhaust all philosophy of science, they form a necessary part of it. They are in fact indispensable in any truly careful conceptual analysis. One can see this whenever one tries to achieve sufficient clarity in dealing with such philosophy-laden issues as *fundamental stochasticity*, *counterfactual definiteness*, or the nature of random variables. Contextuality turns out to be related to several such issues. Can fundamental stochasticity be conceptually distinguished from *ensemble stochasticity*, and is there a way to demonstrate that one and not the other of these forms of stochasticity is present in a given situation? It turns out that contextuality in combination with another property, called *non-disturbance*, can be a key to answering both these questions affirmatively. Is it possible to assign truth values to counterfactual statements about values of random variables, and if so, can counterfactual definiteness (the statement that the value of a variable would be the same if it were recorded in another context) be distinguished from contextuality? Again, a careful conceptual analysis allows one to answer yes to both these questions (with the clarification that it is not the same variable in different contexts one is dealing with but different variables with the same content). Random variables generate their values (say, *Yes* and *No*) with certain probabilities, but to corroborate this one should be able to observe

one and the same random variable repeatedly. How does one know this is the same variable rather than a new variable with the same distribution in every new observation? A careful analysis of this conceptual problem leads one to the notion of *probabilistic couplings* prominently represented in this book.

Genre of the Book I: A Textbook

This book is almost entirely based on the authors' own published work, which, however, the book cites less than sporadically. In general, the book contains very few literature references, and those it does are mostly of the historical nature. The book is written with a deliberate intent *not* to discuss the complex and plentiful literature related to contextuality. This is not a slight to the important achievements of the researchers in the contextuality field. It is simply that the book is more of a textbook than a scholarly survey, and it follows the style of its genre.

We should qualify this statement: In what sense is this book a textbook? One sense in which it is, is that it presents the material systematically, rigorously, and with no previous knowledge thereof required. It is also comprehensive, but only in that it comprehensively covers the basics of contextuality on an abstract mathematical level. This excludes many specialized and domain-specific topics, those that may be of interest in, or even central to, a substantive area of research, such as computer science, psychology, or quantum physics in which most of the contextuality research has been conducted.¹

Also, this book is a textbook in the sense of being accessible. It is mostly confined to systems with a finite number of random variables, and to variables with a finite number of possible values, mostly, just two. This allows the reader to focus on all conceptual issues without getting into technical details requiring previous knowledge of measure-theoretic constructs (although these are introduced too, as optional reading). The book is written to be accessible to college students or even to advanced high-schoolers with only modest knowledge of mathematics: The basic set-theoretic notions and notation, elementary aspects of probability theory, and, in places, basics of linear algebra. Each chapter is followed by exercises with complete solutions. They provide additional examples and clarifications, and sometimes offer ad-

¹ The reader interested in how contextuality applies to quantum physics can find a wealth of information in the available surveys: Yeong-Cherng Liang, Robert W. Spekkens, & Howard M. Wiseman, *Physics Reports* **506**, 1 (2011), and Costantino Budroni, Adán Cabello, Otfried Gühne, Matthias Kleinmann, & Jan-Åke Larsson, *Review of Modern Physics* **94**, 045007 (2022).

ditional theoretical material. Technical issues and involved proofs are put in special sections marked by asterisks. They may be skipped, although hopefully they will not be by better prepared and more skeptical readers. Proofs are often conducted by means of easily generalizable examples. The latter does not mean, however, that the book compromises on deductive reasoning. Quite the opposite, deductive reasoning permeates the entire book, so that the reader can always see how the issues and results logically follow from a few initial definitions, principles, and conventions.

Genre of the Book II: A Treatise for Experts

We should, however, mention the sense in which this book is *not* a textbook as commonly understood: It does not present a widely agreed upon set of well-established topics and results. I am not sure such a set exists. At the contemporary stage of contextuality studies any attempt to present a systematic theory of contextuality which is not domain-specific is bound to be an original approach. The present attempt is based on the theory dubbed *Contextuality-by-Default*, developed by Janne V. Kujala and myself since 2014. Other colleagues, prominently including Víctor H. Cervantes, joined us in this development at its later stages. One feature of the theory is that random variables are systematically identified not only by their *contents* but also by their *contexts*. This double-identification allows one to stay well within the confines of classical probability theory, or more precisely, classical theory of random variables. In fact, the double-identification is a necessary requirement of this theory. A systematic use of the language of random variables has mathematical advantages over the more common probabilistic treatments of contextuality in the language of events. This is because random variables add to probability spaces the flexibility of functions mapping them into each other. Thus, the assumptions traditionally considered necessary for the derivation of the Bell-type inequalities, such as *outcome determinism* and *factorizability*, are satisfied automatically once the underlying hidden variables are treated as true random variables. Our treatment of random variables, however, puts a greater emphasis than is usual on the situations when random variables do not possess joint distributions. As a consequence, there is a greater emphasis on the theory of *probabilistic couplings*. The book also attaches greater than usual importance to the systems of *dichotomous random variables*, and to the possibility of redefining any system of variables as a system of dichotomous ones. We even consider, without committing to it, the possibility of *radical dichotomism*, according

to which the contextuality status of a system is determinable only after the system is dichotomized.

The greatest deviation from the traditional accounts of contextuality, however, is in that we do not constrain the systems of random variables by the requirement that the distribution of a variable only depends on its content and not on its context. In other words, if two variables answer the same question, we allow their distributions to differ. The assumption that they must be the same is justifiable for some quantum-theoretic systems, but it fails to apply to many others (e.g., those describing successive measurements, or the multiple-slit experiments). Moreover, this constraint almost never holds outside physics, for example, in behavioral and social sciences. Our book shows a principled way of isolating and quantifying contextuality in all such situations. (The reader who is only interested in *undisturbed systems*, those with context-independent distributions, still might benefit from reading this book, both because such systems are prominently discussed in the book as special cases, and because it is always useful to see a concept in a more general mathematical setting.) The book also contains some new, previously unpublished or incompletely published developments, such as the hierarchical contextuality measure and a general theory of dichotomizations in structured spaces. Overall, in spite of its technical accessibility, the material presented in this book is advanced enough to be of interest to scholars from a broad spectrum of disciplines: From mathematics to philosophy to quantum physics to computer science to behavioral and social sciences.

Terminology of the Book

There is no universally or even widely accepted terminology in the contextuality literature. Even most basic notions (such as *signaling*) have multiple terminological variants, and some frequently used terms (such as *realism*) allow for multiple interpretations. The terminology adopted in this book therefore follows the internal logic of the theory being developed rather than literature sources. We begin by calling the set of random variables sharing a context a *bunch* of variables (because they form a single random variable), and we call the set of random variables sharing a content a *connection* of variables (because they relate to each other probabilistically unrelated bunches). Virtually all other terms that are not traditional are derivations from these two: *well-bunched*, *well-connected*, *consistently connected*. Of course, like in most systematic treatments of mathematical subjects, in ours one can find degrees of variation in the use of standard mathematical terms. Thus, the standard term *maximal coupling* acquires its generalized version in *multi-*

maximal coupling. We also introduce the term *agglutinativity* to describe the basic property of random variables: A set of variables is jointly distributed if so are elements of a chain of its overlapping subsets. I believe that with a bit of patience and goodwill the reader will find our terminology thoughtfully designed and, due to its generality, convenient to use.

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Dzhafarov,
Kujala, and
Cervantes

Contextuality in Random Variables

The mathematical essence of contextuality lies in the similarity of random variables answering the same question in different contexts: Contextuality means they are less similar when considered within their respective contexts than when isolated from them.

This book presents a principled way of measuring this similarity and distinguishing two forms of context-dependence: contextuality and disturbance. While applicable across a broad range of disciplines, the concept of contextuality in this book is closest to that in quantum physics, where its special forms – in the absence of disturbance – are known as Bell nonlocality and Kochen–Specker contextuality.

This systematic introduction requires no prior familiarity with the subject and very modest mathematical background. Structured as a textbook, complete with exercises and solutions, it is accessible to a broad readership and suitable for teaching. It will be useful to researchers and students in quantum mechanics, philosophy of science, psychology, computer science, linguistics, and probability theory.

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